Geroch monotonicity and the construction of weak solutions of the inverse mean curvature flow

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Abstract

For surfaces evolving under the inverse mean curvature flow, Geroch observed that the Hawking mass is a Lyapunov function. For weak solutions of the flow, the corresponding monotonicity formula was proved by Huisken and Ilmanen. An analogous formula exists for approximate equations as well, and it provides uniform control of the solutions in certain Sobolev spaces. This helps to construct weak solutions under very weak assumptions on the initial data.

1 Introduction

The inverse mean curvature flow is an evolution of hypersurfaces with normal velocity reciprocal to the mean curvature. We study this flow in a complete, connected Riemannian manifold $(\mathcal{N}, \langle \cdot\,, \cdot \rangle)$ of dimension $n \geq 2$. We assume that $\mathcal{N}$ is not compact. A classical solution then consists of an $(n-1)$-dimensional manifold $\mathcal{M}$ and a one-parameter family of embeddings $F(\,\cdot\,, t) : \mathcal{M} \to \mathcal{N}$, with t in an inverval $[0, T)$, satisfying the equation

$$\frac{\partial F}{\partial t} = \frac{\nu}{H} \quad \text{in } \mathcal{M} \times (0, T).$$

Here $H(\,\cdot\,, t)$ and $\nu(\,\cdot\,, t)$ are the mean curvature and the exterior normal vector, respectively, of $M_t = F(\mathcal{M}, t)$. This is a parabolic equation and therefore it is natural to complement it with an initial condition of the form

$$F(\mathcal{M}, 0) = M_0$$

for a given hypersurface $M_0 \subset \mathcal{N}$. In certain situations there are nice existence results for this problem. For example, Gerhardt [2] showed that in a Euclidean space, a classical solution exists for all times if M_0 is the smooth boundary of a bounded, star-shaped set with positive mean curvature. Furthermore, this solution approaches an expanding spherical solution as $t \to \infty$. For other initial data, however, classical solutions may not exist.

A notion of weak solutions, based on a level set formulation, was introduced by Huisken and Ilmanen [5]. The underlying idea is to consider a function u on

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a domain $\Omega \subset \mathcal{N}$ with level sets $M_t = u^{-1}(\{t\})$ evolving by the inverse mean curvature flow. If u is smooth and $\nabla u \neq 0$, then the mean curvature and the normal vector of M_t are

$$H = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{and} \quad \nu = \frac{\nabla u}{|\nabla u|}.$$

Furthermore, the level sets evolve with velocity $1/|\nabla u|$. Hence u gives rise to a solution of the inverse mean curvature flow if, and only if,

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| \quad \text{in } \Omega. \tag{1}$$

Initial data are transformed into boundary data by this approach. If Ω is chosen such that $M_0 = \partial\Omega$, then we need to impose the condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

If M_0 is bounded, then we expect that the flow will expand the surface. Thus in a situation where $\mathcal{N}$ is divided into a bounded and an unbounded component by M_0, the appropriate choice for Ω is the unbounded part. From now on, we assume that $\Omega = \mathcal{N} \backslash E$ for a compact set $E \subset \mathcal{N}$.

Because of the degeneracy of the equation, it is not obvious how weak solutions are best defined. Huisken and Ilmanen used a variational principle. We use the same notion, but we consider a larger function space. Let $\mathrm{BV}_{\mathrm{loc}}(\mathcal{N})$ be the space of all functions $u \in L^1_{\mathrm{loc}}(\mathcal{N})$ with a distributional derivative represented by a $T\mathcal{N}$-valued Radon measure Du. We write $|Du|$ for the total variation of Du.

Definition 1.1. *A function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ is a* weak solution *of* (1) *if for every compact set $K \subset \Omega$ and for every $v \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ with $v = u$ in $\Omega \backslash K$, the inequality*

$$|Du|(K) + \int_K u \, d|Du| \leq |Dv|(K) + \int_K v \, d|Du|$$

holds true. A weak solution is called proper *if*

$$u(x) \to \infty \quad \text{as } \operatorname{dist}(x, E) \to \infty.$$

The definition requires that u minimizes a certain functional—depending on u itself—and (1) is the formal Euler-Lagrange equation for the resulting variational problem. If u is continuous up to the boundary $\partial\Omega$, then we can make sense of the boundary condition as well in this framework. The concept of a proper weak solution provides control of u at infinity; geometrically it means that solutions stay bounded at finite times. Huisken and Ilmanen [5] showed that weak solutions of the inverse mean curvature flow satisfy a comparison principle and that proper weak solutions satisfying the boundary conditions are unique. (Their results were adapted to the somewhat more general formulation of Definition 1.1 by the author [9]). It has been pointed out by Kotschwar and Ni [7], however, that some manifolds do not admit a proper solution.

Under certain assumptions on the geometry of $\mathcal{N}$ and the regularity of $\partial\Omega$, Huisken and Ilmanen also constructed proper weak solutions. The development of this theory was motivated by a property of the inverse mean curvature flow that makes it a valuable tool for a problem in general relativity. The crucial observation in this context is that a certain functional involving the L^2-norm of the mean curvature is a Lyapunov function under the inverse mean curvature flow. Let σ denote the $(n-1)$-dimensional Hausdorff measure in $\mathcal{N}$. For a smooth solution of the inverse mean curvature flow, let $A(\,\cdot\,,t)$ denote the second fundamental form of $M_t = F(\mathcal{M},t)$ and let $D^\top$ denote the gradient on M_t. Then we compute

$$\frac{d}{dt}\int_{M_t} H^2\,d\sigma = -\int_{M_t}\left(2\frac{|D^\top H|^2}{H^2} + 2|A|^2 - H^2 + 2\operatorname{Ric}(\nu,\nu)\right)d\sigma, \tag{3}$$

where Ric denotes the Ricci curvature of $\mathcal{N}$. Now suppose that $n=3$ and the scalar curvature R of $\mathcal{N}$ is non-negative. If $\mathcal{M}$ is a topological sphere, then with the help of the Gauss equation and the Gauss-Bonnet formula, we can derive a monotonicity formula for the so-called Hawking mass

$$m(M) = \sqrt{\frac{\sigma(M)}{64\pi^3}}\left(16\pi - \int_M H^2\,d\sigma\right).$$

Indeed, if λ_1, λ_2 and K are the principal curvatures and the Gauss curvature, respectively, of a closed surface M in $\mathcal{N}$ with Euler characteristic 2, then we have

$$\begin{aligned}&\int_M (-2|A|^2 + H^2 - 2\operatorname{Ric}(\nu,\nu))\,d\sigma\\ &\qquad = \int_M\left(-\frac{1}{2}H^2 - \frac{1}{2}(\lambda_1-\lambda_2)^2 + 2K - R\right)d\sigma \le 8\pi - \frac{1}{2}\int_M H^2\,d\sigma.\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\left(e^{t/2}\left(16\pi - \int_{M_t} H^2\,d\sigma\right)\right) \ge 0.$$

In addition, we compute (in any dimension)

$$\frac{d}{dt}\sigma(M_t) = \sigma(M_t).$$

Hence $\sigma(M_t) = e^t\sigma(M_0)$, and the above inequality implies that $m(M_t)$ is non-decreasing. This observation was made by Geroch [3] and proved by Huisken and Ilmanen for the weak solutions constructed by their method, using an approach based on an elliptic regularization of equation (1).

For $n \neq 3$, we still obtain a similar formula for smooth solutions, although without the physical interpretation. Note that (3) implies

$$\frac{d}{dt}\left(e^{-t}\int_{M_t} H^2\,d\sigma\right) \le -2e^{-t}\int_{M_t}(|A|^2 + \operatorname{Ric}(\nu,\nu))\,d\sigma.$$

In particular, in the case of a non-negative Ricci curvature, the average square mean curvature is non-increasing. Furthermore, similar computations yield

$$\frac{d}{dt}\left(e^{-t}\int_{M_t} H^{q+1}\,d\sigma\right) \le -(q+1)e^{-t}\int_{M_t} H^{q-1}(|A|^2 + \operatorname{Ric}(\nu,\nu))\,d\sigma$$

for every $q \geq 1$. We use the expression 'Geroch monotonicity' for any of these inequalities.

In this paper we study the role of Geroch monotonicity in the context of a specific approach to the construction of weak solutions of the inverse mean curvature flow, introduced by the author [8, 9] and extended by Kotschwar and Ni [7]. This method is based on an approximation of (1) by the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p \quad \text{in } \Omega \tag{4}$$

for $p > 1$ and the observation that the transformation $v = e^{u/(1-p)}$ gives rise to

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in } \Omega,$$

an equation with a rich existing theory. It turns out that the Geroch monotonicity formulas do not only have a counterpart for $p > 1$ that controls the second fundamental form when we let $p \searrow 1$, but it also allows us to derive estimates for u in certain Sobolev spaces. These inequalities are local in Ω, and therefore they require no assumptions on the regularity of $\partial\Omega$. In the theory below, we use no conditions other than compactness of $E \neq \emptyset$ and $\overline{E^\circ} = E$.

In order to obtain solutions under such weak assumptions, we need to relax the boundary conditions. Even for a Lipschitz regular boundary, the example of a 'blossoming cone' by Huisken and Ilmanen [4] suggests that solutions with a reasonable geometric interpretation need not be continuous on the boundary (even though this example is for an unbounded E and does therefore not fit into the framework discussed here). We replace (2) by the condition that

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = 0$$

for every point $x_0 \in \partial\Omega$. Furthermore, we show that the solutions constructed below are continuous at every boundary point where $\partial\Omega$ is sufficiently regular.

Regularity is defined by an exterior ball condition in this context. For $y_0 \in \mathcal{N}$ and $r > 0$, we use the notation $B_r(y_0)$ for the open geodesic ball in $\mathcal{N}$ with centre y_0 and radius r. Furthermore, we write δ for the distance function on $\mathcal{N}$.

Definition 1.2. *A point $x_0 \in \partial\Omega$ is called* regular *if for every $\epsilon > 0$ there exist a point $y_0 \in E$ and a radius $r > 0$ such that $B_r(x_0) \subset E$ and $\delta(x_0, y_0) \leq r(1+\epsilon)$.*

Before we can state the main result, we also need the notion of a second fundamental form for the level sets of a function in $\mathrm{BV}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$, because this quantity appears in the Geroch monotonicity formula. We use a definition of Huisken and Ilmanen [5], which is also related to a concept introduced by Hutchinson [6].

Consider first a smooth hypersurface $M \subset \mathcal{N}$ with normal vector ν. We extend ν to $\mathcal{N}$ such that $\nabla_\nu \nu = 0$ on M. Now we consider the section $\nabla\nu$ of the vector bundle $\operatorname{End}(T\mathcal{N})$ and we identify the second fundamental form A with its restriction to M. Suppose that we have an orthonormal tangent frame field $(e_1, \ldots, e_n)$ in $\mathcal{N}$. Then for any smooth section P of $\operatorname{End}(T\mathcal{N})$ with compact support, an integration by parts gives

$$\begin{aligned}
\int_M H\,\langle \nu, P\nu \rangle \, d\sigma &= \int_M \left(\sum_{i=1}^n \langle \nabla_{e_i}(P\nu), e_i \rangle - \langle \nabla_\nu(P\nu), \nu \rangle \right) d\sigma \\
&= \int_M \left(\sum_{i=1}^n \langle \nabla_{e_i} P\nu, e_i \rangle + \langle P, A \rangle - \langle \nabla_\nu P\nu, \nu \rangle \right) d\sigma.
\end{aligned}$$

(We have used the symmetry of A in the last step.) If we apply this formula to all the level sets $u^{-1}(\{t\})$ of a certain function u and integrate over t, then we obtain an identity that can be represented in terms of integrals over Ω, using the coarea formula. This is the motivation for the following definition.

Definition 1.3. *For $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, let ν be a unit tangent vector field on Ω such that $Du = |Du| \llcorner \nu$. Suppose that there exists a section A of $\mathrm{End}(T\Omega)$ with locally $|Du|$-integrable coefficients, such that A is symmetric and $A\nu = 0$ at $|Du|$-almost every point in Ω, and for every smooth section P of $\mathrm{End}(T\Omega)$ with compact support,*

$$\int_\Omega \left(\langle \nu, P\nu \rangle \operatorname{tr} A - \langle P, A \rangle - \operatorname{tr}(\nabla P\nu) + \langle \nabla_\nu P\nu, \nu \rangle \right) d|Du| = 0.$$

Then A is called the weak second fundamental form of the level sets *of u.*

It is readily checked that the weak second fundamental form is unique (up to a $|Du|$-null set) if it exists.

If we have a function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, then almost all sublevel sets are of locally finite perimeter. We use the notation $\partial^* G$ for the reduced boundary of a set $G \subset \Omega$ of locally finite perimeter. Quantities as appearing in the Geroch monotonicity formulas can then be represented as integrals over the reduced boundaries of sublevel sets. We regard the monotonicity formulas (and their counterparts for p-harmonic functions) mostly as tools to control the approximate solutions, and it is not clear whether they remain valid in the limit $p \searrow 1$ when $q > 1$. But considering the case $q = 1$, we do get an estimate for the square mean curvature and the second fundamental form in the limit.

We now assume that $E \neq \emptyset$ is compact and $\overline{E^\circ} = E$. We set $\Omega = \mathcal{N} \backslash E$. Let dV be the volume form on $\mathcal{N}$. For every $p \in (1, 2]$, let $\dot{W}^{1,p}(\mathcal{N})$ be the completion of $C_0^\infty(\mathcal{N})$ with respect to the norm

$$\|\phi\|_{\dot{W}^{1,p}(\mathcal{N})} = \left(\int_{\mathcal{N}} |\nabla \phi|^p \, dV \right)^{1/p}.$$

Let $v_p \in \dot{W}^{1,p}(\mathcal{N})$ be a minimizer of the norm in $\dot{W}^{1,p}(\mathcal{N})$ among all $v \in \dot{W}^{1,p}(\mathcal{N})$ with $v \geq 1$ in E. Note that v_p can be identified with a function in $W^{1,p}_{\mathrm{loc}}(\mathcal{N})$, as truncation at a level above 1 or below 0 will decrease the value of the functional. Furthermore, we have $v_p(x) \in [0, 1]$ almost everywhere. We set $u_p = (1 - p) \log v_p$.

Theorem 1.1. *There exist a sequence $p_k \searrow 0$ and a function $u \in \mathrm{BV}_{\mathrm{loc}}(\mathcal{N}) \cap C^0(\Omega)$ such that $u_{p_k} \to u$ locally uniformly in Ω. Moreover, the limit has the following properties.*

(i) It is a weak solution of (1).

(ii) For every point $x_0 \in \partial\Omega$,

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = 0.$$

(iii) For every regular point $x_0 \in \partial\Omega$,

$$\lim_{x \to x_0} u(x) = 0.$$

(iv) It belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$.

(v) The weak second fundamental form A of the level sets of u exists.

(vi) For $t > 0$, set $E_t = u^{-1}([0,t])$. Let

$$T = \lim_{r \searrow 0} \sup_{\operatorname{dist}(x,E) < r} u(x)$$

and $t_0 > T$. If u is proper, then there exists a constant $C > 0$ such that for every $\tau > t_0$,

$$e^{-\tau} \int_{\partial^* E_\tau} |\nabla u|^2 \, d\sigma + 2 \int_{t_0}^{\tau} e^{-t} \int_{\partial^* E_t} |A|^2 \, d\sigma \, dt \leq C.$$

Remarks. (i) The theory of Kotschwar and Ni [7] provides criteria under which u is proper.

(ii) Uniqueness is not clear even if u is proper. Unless all boundary points are regular, the lack of continuity at the boundary prevents a direct application of the comparison principle.

(iii) If Ω has a Lipschitz boundary, then it follows from Rademacher's theorem that σ-almost every boundary point is regular. The boundary condition $u = 0$ on $\partial\Omega$ is then satisfied in the sense of traces [1, Sect. 3.8].

(iv) The number T can be interpreted as the time when the generalized hypersurface $\partial^* E_t$ detaches from $\partial\Omega$.

The proof of Theorem 1.1 is based on an analysis of equation (4) and the behaviour of its solutions as $p \searrow 1$. In the next section, we derive an inequality that can be regarded as a version of the Geroch monotonicity formula for $p > 1$ and we use it to prove estimates for solutions of (4) in $W^{1,q}_{\mathrm{loc}}(\Omega)$. Then we discuss the notion of a measure-section pair, which is an adaption of the idea of measure-function pairs introduced by Hutchinson [6]. We need this concept to control the second fundamental form when we let $p \searrow 1$. In the final section, we study this limit and prove the theorem.

The statement $u \in \bigcap_{q<\infty} W^{1,q}_{\mathrm{loc}}(\Omega)$ can be improved to $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ if the arguments in the proof of the theorem are combined with an estimate of Kotschwar and Ni [7, Theorem 1.1]. Indeed, the statements from (i)–(iv) follow from their results with a few easy arguments (which can be found in section 4). We prefer to use a different proof, however, which highlights the connection between regularity and Geroch monotonicity. The statements (v) and (vi) are new under the conditions of the theorem.

2 Estimates for p-harmonic functions

In this section we derive a version of the Geroch monotonicity formula for solutions of the equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^p \quad \text{in } \Omega \tag{5}$$

and we use it to find L^q-estimates for $|\nabla u|$. We will need the results for the proof of Theorem 1.1, but we obtain estimates for p-harmonic functions as well, which may be of independent interest. We use only local arguments in this section, and thus we may replace Ω by any open subset of $\mathcal{N}$ if we wish.

Let $p \in (1, 2]$. Consider a solution $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in } \Omega \tag{6}$$

that is positive and bounded. We may rescale if necessary, and thus we use the assumption

$$0 < v \leq 1 \quad \text{in } \Omega.$$

Solutions of the variational problem in the introduction, of course, satisfy the condition automatically. The function $u = (1-p)\log v$ then satisfies equation (5) and $u \geq 0$.

It is easy to obtain local estimates for the L^p-norms of $|\nabla v|$ and $|\nabla u|$. Indeed, for any $\eta \in C_0^\infty(\Omega)$ with $\eta \geq 0$, we have

$$\begin{aligned}\int_\Omega \eta^p |\nabla v|^p \, dV &= -p \int_\Omega \eta^{p-1} v |\nabla v|^{p-2} \langle \nabla\eta, \nabla v\rangle \, dV \\ &\leq p \left(\int_\Omega \eta^p |\nabla v|^p \, dV\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla \eta|^p \, dV\right)^{\frac{1}{p}}.\end{aligned}$$

Thus

$$\int_\Omega \eta^p |\nabla v|^p \, dV \leq p^p \int_\Omega |\nabla\eta|^p \, dV.$$

Furthermore,

$$\int_\Omega \eta^p |\nabla u|^p \, dV = -p \int_\Omega \eta^{p-1} |\nabla u|^{p-2} \langle \nabla\eta, \nabla u\rangle \, dV,$$

and using Hölder's inequality again, we see that

$$\int_\Omega \eta^p |\nabla u|^p \, dV \leq p^p \int_\Omega |\nabla\eta|^p \, dV$$

as well.

There are good regularity results for p-harmonic functions [10]. In particular, it is known that ∇v is Hölder continuous and v is smooth away from the set $\{x \in \Omega \colon \nabla v(x) = 0\}$. But since most works do not study the dependence of the corresponding inequalities on p explicitly, we need to re-examine the regularity. In order to formulate the results concisely, we introduce some notation. For a differentiable function $f : \Omega \to \mathbb{R}$, we define

$$D^\perp f = \frac{\langle \nabla f, \nabla u\rangle}{|\nabla u|} \quad \text{and} \quad D^\top f = \nabla f - D^\perp f \frac{\nabla u}{|\nabla u|}$$

where $\nabla u \neq 0$, and $D^\perp = 0$, $D^\top = \nabla$ where $\nabla u = 0$. For two tangent vector fields X, Y on Ω,

$$D^\perp X = \frac{\nabla_{\nabla u} X}{|\nabla u|} \quad \text{and} \quad D^\top_Y X = \nabla_Y X - D^\perp X \frac{\langle Y, \nabla u\rangle}{|\nabla u|}$$

(similarly extended to points where $\nabla u = 0$). That is, we decompose the gradient and the covariant derivative into the parts perpendicular and tangential, respectively, to the level sets of u. We wish to prove the following inequality.

Proposition 2.1. *Let $q \geq p$ and define*

$$c_1 = \frac{q-p+2}{p}, \quad c_2 = (q-1)c_1, \quad \text{and} \quad c_3 = (p-1)(q-p+1)c_1.$$

Let $\eta \in C_0^\infty(\Omega)$ and suppose that K is a constant with $\mathrm{Ric} \geq -K \langle \cdot\,, \cdot \rangle$ in $\operatorname{supp}\eta$. If $p \leq 1 + \frac{1}{16c_1}$, then

$$\int_\Omega \eta^2 e^{-2u} |\nabla u|^{q-2} \left(|\nabla u|^4 + c_1 |D^\top \nabla u|^2 + c_2 |D^\top D^\perp u|^2 + c_3 |D^\perp D^\perp u|^2 \right) dV$$
$$\leq (9 + 4c_1) \int_\Omega e^{-2u} |\nabla \eta|^2 |\nabla u|^q \, dV + 2c_1 K \int_\Omega \eta^2 e^{-2u} |\nabla u|^q \, dV.$$

We now fix an orthonormal tangent frame field $(e_1, \ldots, e_n)$ on $\mathcal{N}$. It need not be continuous, so there is no question about its existence. In the proof of the proposition we use the following observation.

Lemma 2.1. *Let X, Y be smooth tangent vector fields. Then*

$$\langle X, \nabla \operatorname{div} Y \rangle = \operatorname{div}(\nabla_X Y) - \sum_{i=1}^n \left\langle \nabla_{\nabla_{e_i} X} Y, e_i \right\rangle - \mathrm{Ric}(X, Y).$$

Proof. Let $x_0 \in \mathcal{N}$. When we evaluate the right hand side of the formula at x_0, then the values of e_i away from x_0 do not matter. Thus we may assume that e_i is smooth in a neighbourhood of x_0 and $\nabla e_i(x_0) = 0$ for $i = 1, \ldots, n$. Let Rm denote the Riemann curvature tensor.

Now at x_0, we have

$$\begin{aligned}
\langle X, \nabla \operatorname{div} Y \rangle = X(\operatorname{div} Y) &= \sum_{i=1}^n X \langle \nabla_{e_i} Y, e_i \rangle = \sum_{i=1}^n \langle \nabla_X \nabla_{e_i} Y, e_i \rangle \\
&= \sum_{i=1}^n \left(\langle \nabla_{e_i} \nabla_X Y, e_i \rangle + \left\langle \nabla_{[X, e_i]} Y, e_i \right\rangle + \langle \mathrm{Rm}(X, e_i) Y, e_i \rangle \right) \\
&= \operatorname{div}(\nabla_X Z) - \sum_{i=1}^n \left\langle \nabla_{\nabla_{e_i} X} Y, e_i \right\rangle - \mathrm{Ric}(X, Y),
\end{aligned}$$

using in last step the observation that $[X, e_i] = -\nabla_{e_i} X$ at x_0, as the Levi-Civita connection is torsion free. □

Proof of Proposition 2.1. We first approximate v by solutions of a regularized problem. Choose a bounded, open set $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega$. For $\epsilon > 0$, let $v_\epsilon \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional

$$F_\epsilon^p(w) = \frac{1}{p} \int_{\Omega'} (|\nabla w|^2 + \epsilon^2)^{p/2} \, dV$$

among all $w \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $w = v$ almost everywhere outside of Ω'. We use the abbreviation

$$a_\epsilon = (|\nabla v_\epsilon|^2 + \epsilon^2)^{1/2}.$$

Then

$$\operatorname{div}(a_\epsilon^{p-2} \nabla v_\epsilon) = 0 \quad \text{in } \Omega'. \tag{7}$$

Obviously, we have $F_\epsilon^p(v_\epsilon) \leq F_\epsilon^p(v)$, and therefore we have a family of functions that is bounded in $W^{1,p}(\Omega')$. Furthermore, standard elliptic theory implies that v_ϵ is smooth.

The theory of Tolksdorf [10] gives further local bounds for the derivatives of v_ϵ that are uniform in ϵ: for every precompact set $\Omega'' \subset \Omega'$ there exist constants $\alpha \in (0,1]$ and $C > 0$ such that for every $\epsilon \in (0,1]$,

$$\|\nabla^2 v_\epsilon\|_{L^p(\Omega'')} + \|\nabla v_\epsilon\|_{C^{1,\alpha}(\overline{\Omega''})} \leq C.$$

Thus there exists a sequence $\epsilon_k \searrow 0$ such that v_{ϵ_k} converges weakly in $W^{2,p}(\Omega'')$ and strongly in $C^1(\overline{\Omega''})$ for any such Ω''. The limit is p-harmonic, and because p-harmonic functions are subject to a comparison principle [11], the limit is v. Thus we have in fact $v_\epsilon \to v$ in the above sense as $\epsilon \searrow 0$.

Set $u_\epsilon = (1-p)\log v_\epsilon$. As v is continuous, there exists a number $s > 0$ such that $v \geq s$ in Ω'. Applying the maximum principle to equation (7), we obtain $v_\epsilon \geq s$ as well, and it follows that $u_\epsilon \to u$ as $\epsilon \searrow 0$ weakly in $W^{2,p}(\Omega'')$ and strongly in $C^1(\overline{\Omega''})$ for every precompact open set $\Omega'' \subset \Omega'$. We now compute

$$a_\epsilon^{p-2}\nabla v_\epsilon = (1-p)^{1-p} e^{-u_\epsilon}\left(\left(|\nabla u_\epsilon|^2 + (p-1)^2\epsilon^2 v_\epsilon^{-2}\right)^{p/2-1}\nabla u_\epsilon\right).$$

Set

$$b_\epsilon = \left(|\nabla u_\epsilon|^2 + (p-1)^2\epsilon^2 v_\epsilon^{-2}\right)^{1/2}.$$

Then we have

$$0 = \operatorname{div}(e^{-u_\epsilon} b_\epsilon^{p-2}\nabla u_\epsilon) = e^{-u_\epsilon}\left(\operatorname{div}(b_\epsilon^{p-2}\nabla u_\epsilon) - b_\epsilon^{p-2}|\nabla u_\epsilon|^2\right).$$

That is,

$$\operatorname{div}(b_\epsilon^{p-2}\nabla u_\epsilon) = b_\epsilon^{p-2}|\nabla u_\epsilon|^2 \quad \text{in } \Omega'.$$

We compute

$$\begin{aligned}
\operatorname{div}(b_\epsilon^q \nabla u_\epsilon) &= b_\epsilon^q|\nabla u_\epsilon|^2 + b_\epsilon^{p-2}\left\langle \nabla u_\epsilon, \nabla b_\epsilon^{q-p+2}\right\rangle \\
&= b_\epsilon^q|\nabla u_\epsilon|^2 + c_1 b_\epsilon^{q-p}\left\langle \nabla u_\epsilon, \nabla b_\epsilon^{p}\right\rangle \\
&= b_\epsilon^q|\nabla u_\epsilon|^2 + c_1 b_\epsilon^{q-p}\left\langle \nabla u_\epsilon, \nabla \operatorname{div}(b_\epsilon^{p-2}\nabla u_\epsilon)\right\rangle \\
&\quad + c_1(p-1)^2\epsilon^2 b_\epsilon^{q-p}\left\langle \nabla u_\epsilon, \nabla\left(b_\epsilon^{p-2}v_\epsilon^{-2}\right)\right\rangle \\
&= b_\epsilon^q|\nabla u_\epsilon|^2 + c_1 \operatorname{div}\left(b_\epsilon^{q-p}\nabla_{\nabla u_\epsilon}(b_\epsilon^{p-2}\nabla u_\epsilon)\right) \\
&\quad - c_1\sum_{i=1}^n \left\langle \nabla_{\nabla_{e_i}(b_\epsilon^{q-p}\nabla u_\epsilon)}(b_\epsilon^{p-2}\nabla u_\epsilon), e_i\right\rangle - c_1 b_\epsilon^{q-2}\operatorname{Ric}(\nabla u_\epsilon, \nabla u_\epsilon) \\
&\quad + c_1(p-1)^2\epsilon^2 b_\epsilon^{q-p}\left\langle \nabla u_\epsilon, \nabla\left(b_\epsilon^{p-2}v_\epsilon^{-2}\right)\right\rangle,
\end{aligned}$$

using Lemma 2.1 in the last step. We write

$$D_\epsilon^\perp f = \frac{\langle \nabla f, \nabla u_\epsilon\rangle}{|\nabla u_\epsilon|} \quad \text{and} \quad D_\epsilon^\top f = \nabla f - D_\epsilon^\perp f \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}$$

for a function f and

$$D_\epsilon^\perp X = \frac{\nabla_{\nabla u_\epsilon} X}{|\nabla u_\epsilon|} \quad \text{and} \quad D_\epsilon^\top X = \nabla X - \frac{du_\epsilon}{|\nabla u_\epsilon|} \otimes D_\epsilon^\perp X$$

for a vector field X. Note that

$$\nabla b_\epsilon = \frac{1}{b_\epsilon}\left(|\nabla u_\epsilon| D_\epsilon^\perp \nabla u_\epsilon + 2(p-1)\epsilon^2 v_\epsilon^{-2}\nabla u_\epsilon\right).$$

Thus

$$\begin{aligned} b_\epsilon^{q-p}\nabla_{\nabla u_\epsilon}(b_\epsilon^{p-2}\nabla u_\epsilon) = {} & b_\epsilon^{q-2}|\nabla u_\epsilon| D_\epsilon^\perp \nabla u_\epsilon \\ & + (p-2) b_\epsilon^{q-4}|\nabla u_\epsilon|^2 (D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)\nabla u_\epsilon \\ & + 2(p-2)(p-1)\epsilon^2 v_\epsilon^{-2} b_\epsilon^{q-4}|\nabla u_\epsilon|^2 \nabla u_\epsilon. \end{aligned}$$

Furthermore,

$$\begin{aligned} \sum_{i=1}^n \left\langle \nabla_{\nabla_{e_i}(b_\epsilon^{q-p}\nabla u_\epsilon)}(b_\epsilon^{p-2}\nabla u_\epsilon), e_i\right\rangle = {} & b_\epsilon^{q-2}|\nabla^2 u_\epsilon|^2 + (q-2) b_\epsilon^{q-4}|\nabla u_\epsilon|^2 |D_\epsilon^\perp \nabla u_\epsilon|^2 \\ & + (q-p)(p-2) b_\epsilon^{q-6}|\nabla u_\epsilon|^4 (D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)^2 \\ & + 2(q-2)(p-1)\epsilon^2 b_\epsilon^{q-4} v_\epsilon^{-2}|\nabla u_\epsilon|^2 D_\epsilon^\perp D_\epsilon^\perp u_\epsilon \\ & + 4(q-p)(p-2)(p-1)^2 \epsilon^4 b_\epsilon^{q-6} v_\epsilon^{-4}|\nabla u_\epsilon|^4. \end{aligned}$$

We now consider the limit $\epsilon \searrow 0$ in these formulas. We conclude that there exists a distribution g_ϵ with $g_\epsilon \to 0$ in $(W_0^{1,\infty}(\Omega''))^*$ such that

$$\begin{aligned} & \operatorname{div}\left(|\nabla u_\epsilon|^q \nabla u_\epsilon - c_1 |\nabla u_\epsilon|^{q-1} D_\epsilon^\perp \nabla u_\epsilon - c_1(p-2)|\nabla u_\epsilon|^{q-2} D_\epsilon^\perp D_\epsilon^\perp u_\epsilon \nabla u_\epsilon\right) \\ & \quad \le g_\epsilon + |\nabla u_\epsilon|^{q+2} - c_1 |\nabla u_\epsilon|^{q-2}\big(|\nabla^2 u_\epsilon|^2 + (q-2)|D_\epsilon^\perp \nabla u_\epsilon|^2 \\ & \qquad\qquad + (q-p)(p-2)(D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)^2 + \operatorname{Ric}(\nabla u_\epsilon, \nabla u_\epsilon)\big) \\ & \quad = g_\epsilon + |\nabla u_\epsilon|^{q+2} - c_1 |\nabla u_\epsilon|^{q-2}\big(|D_\epsilon^\top \nabla u_\epsilon|^2 + (q-1)|D_\epsilon^\top D_\epsilon^\perp u_\epsilon|^2 \\ & \qquad\qquad + (p-1)(q-p+1)(D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)^2 + \operatorname{Ric}(\nabla u_\epsilon, \nabla u_\epsilon)\big) \end{aligned} \tag{8}$$

In the last step we have used the fact that

$$|\nabla^2 u_\epsilon|^2 = |D_\epsilon^\top \nabla u_\epsilon|^2 + |D_\epsilon^\perp \nabla u_\epsilon|^2$$

and

$$|D_\epsilon^\perp \nabla u_\epsilon|^2 = |D_\epsilon^\top D_\epsilon^\perp u_\epsilon|^2 + |D_\epsilon^\perp D_\epsilon^\perp u_\epsilon|^2;$$

therefore

$$\begin{aligned} & |\nabla^2 u_\epsilon|^2 + (q-2)|D_\epsilon^\perp \nabla u_\epsilon|^2 + (q-p)(p-2)(D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)^2 \\ & \qquad = |D_\epsilon^\top \nabla u_\epsilon|^2 + (q-1)|D_\epsilon^\top D_\epsilon^\perp u_\epsilon|^2 + (p-1)(q-p+1)(D_\epsilon^\perp D_\epsilon^\perp u_\epsilon)^2. \end{aligned}$$

We know that $|\nabla u_\epsilon|^{q/2-1} D_\epsilon^\top \nabla u_\epsilon$ converges weakly in $L^p_{\mathrm{loc}}(\Omega')$ to the limit $|\nabla u|^{q/2-1} D^\top \nabla u$, and we have similar convergence for the expressions involving $D_\epsilon^\top D_\epsilon^\perp u_\epsilon$ and $D_\epsilon^\perp D_\epsilon^\perp u_\epsilon$. Inequality (8) then gives a local L^2-bound for these functions, and we conclude that we have weak convergence in $L^2_{\mathrm{loc}}(\Omega')$ as well. Passing to the limit and using the notation

$$F^2 = |D^\top \nabla u|^2 + (q-1)|D^\top D^\perp u|^2 + (p-1)(q-p+1)(D^\perp D^\perp u)^2,$$

we obtain

$$\begin{aligned} & \operatorname{div}\left(|\nabla u|^q \nabla u - c_1 |\nabla u|^{q-1} D^\top D^\perp u - c_1(p-1)|\nabla u|^{q-2} D^\perp D^\perp u \nabla u\right) \\ & \qquad\qquad \le |\nabla u|^{q+2} - c_1 |\nabla u|^{q-2}(F^2 + \operatorname{Ric}(\nabla u, \nabla u)). \end{aligned} \tag{9}$$

Thus we also have

$$\begin{aligned}\operatorname{div}\left(e^{-2u}\left(|\nabla u|^q\nabla u - c_1|\nabla u|^{q-1}D^\top D^\perp u - c_1(p-1)|\nabla u|^{q-2}D^\perp D^\perp u\nabla u\right)\right)& \\ \le c_1 e^{-2u}|\nabla u|^{q-2}\left(2(p-1)|\nabla u|^2 D^\perp D^\perp u - F^2 - \operatorname{Ric}(\nabla u,\nabla u)\right)& \\ - e^{-2u}|\nabla u|^{q+2}.&\end{aligned}$$

Let $\eta \in C_0^\infty(\Omega)$. Testing the last inequality with η^2, we obtain

$$\begin{aligned}&\int_\Omega \eta^2 e^{-2u}\left(|\nabla u|^{q+2} + c_1|\nabla u|^{q-2}F^2\right)\,dV\\ &\quad\le 2c_1(p-1)\int_\Omega \eta^2 e^{-2u}|\nabla u|^q D^\perp D^\perp u\,dV + 2\int_\Omega \eta e^{-2u}|\nabla u|^q\langle\nabla\eta,\nabla u\rangle\,dV\\ &\qquad - 2c_1\int_\Omega \eta e^{-2u}|\nabla u|^{q-1}\left\langle \nabla\eta, D^\top D^\perp u + (p-1)D^\perp D^\perp u\frac{\nabla u}{|\nabla u|}\right\rangle dV\\ &\qquad - c_1\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q-2}\operatorname{Ric}(\nabla u,\nabla u)\,dV.\end{aligned}$$

We now estimate the terms on the right hand side one by one using Young's inequality. We have

$$\begin{aligned}2c_1(p-1)\int_\Omega \eta^2 e^{-2u}|\nabla u|^q D^\perp D^\perp u\,dV &\le 4c_1(p-1)\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q+2}\,dV\\ &\quad + \frac{c_1}{4}(p-1)\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q-2}(D^\perp D^\perp u)^2\,dV\end{aligned}$$

and

$$\begin{aligned}2\int_\Omega \eta e^{-2u}|\nabla u|^q\langle\nabla\eta,\nabla u\rangle\,dV &\le \frac{1}{4}\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q+2}\,dV\\ &\quad + 4\int_\Omega e^{-2u}|\nabla\eta|^2|\nabla u|^q\,dV\end{aligned}$$

and

$$\begin{aligned}-2c_1\int_\Omega \eta e^{-2u}|\nabla u|^{q-1}\langle\nabla\eta, D^\top D^\perp u\rangle\,dV &\le \frac{c_1}{2}\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q-2}|D^\top\nabla u|^2\,dV\\ &\quad + 2c_1\int_\Omega e^{-2u}|\nabla\eta|^2|\nabla u|^q\,dV\end{aligned}$$

and

$$\begin{aligned}&-2c_1(p-1)\int_\Omega \eta e^{-2u}|\nabla u|^{q-2}\langle\nabla\eta,\nabla u\rangle D^\perp D^\perp u\,dV\\ &\qquad\le \frac{c_1}{4}(p-1)\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q-2}(D^\perp D^\perp u)^2\,dV\\ &\qquad\quad + 4c_1(p-1)\int_\Omega e^{-2u}|\nabla\eta|^2|\nabla u|^q\,dV\end{aligned}$$

and

$$-c_1\int_\Omega \eta^2 e^{-2u}|\nabla u|^{q-2}\operatorname{Ric}(\nabla u,\nabla u)\,dV \le c_1 K\int_\Omega \eta^2 e^{-2u}|\nabla u|^q\,dV.$$

Now suppose that $p \le 1 + \frac{1}{16c_1}$. Then it follows that

$$\begin{aligned}\frac{1}{2}\int_\Omega \eta^2 e^{-2u}\left(|\nabla u|^{q+2} + c_1|\nabla u|^{q-2}F^2\right)\,dV \\ \le \left(\frac{17}{4}+2c_1\right)\int_\Omega e^{-2u}|\nabla\eta|^2|\nabla u|^q\,dV + c_1K\int_\Omega \eta^2 e^{-2u}|\nabla u|^q\,dV.\end{aligned}$$

This implies the desired inequality. □

We conclude this section with a few other remarks about the inequality of Proposition 2.1. As u is in $W^{2,p}_{\mathrm{loc}}(\Omega)$ and smooth away from the zeroes of ∇u, it is readily checked that the second fundamental form of its level sets is given by the orthogonal projection of $D^\top\nabla u/|\nabla u|$ onto the tangent space of the level sets. That is,

$$A = \frac{D^\top\nabla u}{|\nabla u|} - D^\top D^\perp u \otimes \frac{\nabla u}{|\nabla u|^2}. \tag{10}$$

Thus Proposition 2.1 gives an estimate for

$$c_1\int_\Omega \eta^2 e^{-2u}|\nabla u|^q|A|^2\,dV.$$

Furthermore, inequality (9) implies

$$\begin{aligned}\operatorname{div}\left(|\nabla u|^q\nabla u - c_1|\nabla u|^{q-1}D^\top D^\perp u - c_1(p-1)|\nabla u|^{q-2}D^\perp D^\perp u\nabla u\right) \\ \le |\nabla u|^{q+2} - c_1|\nabla u|^q|A|^2 - c_1|\nabla u|^{q-2}\operatorname{Ric}(\nabla u,\nabla u).\end{aligned}$$

Let $t_1 > t_0 > 0$ such that $u^{-1}([t_0,t_1]) \subset \Omega$ is bounded. Choose $\psi \in C_0^\infty(t_0,t_1)$ with $\psi \ge 0$. Then we can test the inequality with $e^{-u}\psi(u)$ and we obtain

$$\begin{aligned}c_1\int_\Omega \psi(u)e^{-u}\left(|\nabla u|^q|A|^2 + (p-1)|\nabla u|^q D^\perp D^\perp u + |\nabla u|^{q-2}\operatorname{Ric}(\nabla u,\nabla u)\right)\,dV \\ \le \int_\Omega \psi'(u)e^{-u}|\nabla u|^q\left(|\nabla u|^2 - c_1(p-1)D^\perp D^\perp u\right)\,dV. \quad (11)\end{aligned}$$

This inequality can be regarded as a version of the Geroch monotonicity formula for $p > 1$.

3 Measure-section pairs

We now discuss a tool that we will need to control the weak second fundamental forms of the level sets when we let $p \searrow 0$. It is based on the theory of measure-function pairs developed by Hutchinson [6], but we have to work with the sections of certain vector bundles instead of functions. In this section, we assume that $\Omega \subset \mathcal{N}$ is any open set, not necessarily with a compact complement. Let $\varpi : W \to \Omega$ be a vector bundle over Ω with bundle metric $\langle\cdot,\cdot\rangle$. We also fix a point $x_0 \in \Omega$.

Definition 3.1. *A* measure-section pair *over Ω with values in W is a pair (μ, f), where μ is a Radon measure on Ω and f is a section of W with coefficients in $L^1_{\mathrm{loc}}(\mu)$.*

Suppose that $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and A is the weak second fundamental form of its level sets. Then $(|Du|, A)$ is an example of a measure-section pair with values in $\mathrm{End}(T\Omega)$, and this is the reason why we consider the concept.

Definition 3.2. *Let $p \in [1, \infty)$. Let (μ_k, f_k), $k \in \mathbb{N}$, and (μ, f) be measure-section pairs over Ω with values in W such that $|f_k| \in L^p(\mu_k)$ for every k and $|f| \in L^p(\mu)$. We say that (μ_k, f_k)* converges L^p-weakly *to (μ, f) if*

- *$\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in $(C_0^0(\Omega))^*$,*
- *for every continuous section ϕ of W with compact support,*

$$\int_\Omega \langle f_k, \phi \rangle \, d\mu_k \to \int_\Omega \langle f, \phi \rangle \, d\mu,$$

  and

- *the norms $\|f_k\|_{L^p(\mu_k)}$ are uniformly bounded.*

We say that the convergence is L^p-strong *if*

- *for all $\psi \in C_0^0(W)$,*

$$\lim_{k \to \infty} \int_\Omega \psi(f_k) \, d\mu_k = \int_\Omega \psi(f) \, d\mu,$$

  and

- *the sets $S_{kj} = \{x \in \Omega \colon \delta(x, x_0) \geq j \text{ or } |f_k(x)| \geq j\}$ satisfy*

$$\lim_{j \to \infty} \int_{S_{jk}} |f_k|^p \, d\mu_k = 0$$

  uniformly in k.

Remark. Note that δ is still the distance function in $\mathcal{N}$, not in Ω. As $\mathcal{N}$ is connected and complete, the definition is independent of the choice of x_0.

This is a generalization of weak and strong L^p-convergence for a fixed measure. The following was proved by Hutchinson [6] in the case of a trivial bundle. The general case is reduced to his results with the help of local coordinates and a partition of unity.

Theorem 3.1. *Let $1 < p < \infty$. For $k \in \mathbb{N}$, let (μ_k, f_k) be measure-section pairs over Ω with values in W.*

(i) If for every compact set $K \subset \Omega$,

$$\sup_{k \in \mathbb{N}} \left(\mu_k(K) + \int_\Omega |f_k|^p \, d\mu_k \right) < \infty,$$

then there exists a subsequence that converges L^p-weakly.

(ii) Let (μ, f) be a measure-section pair over Ω with values in W such that (μ_k, f_k) converges L^p-weakly to (μ, f). Then

$$\|f\|_{L^p(\mu)} \leq \liminf_{k \to \infty} \|f_k\|_{L^p(\mu_k)}.$$

(iii) If $\|f\|_{L^p(\mu)} = \lim_{k\to\infty} \|f_k\|_{L^p(\mu_k)}$ in the preceding statement, then the convergence is L^p-strong.

When we work with measure-section pairs, then the following notion is convenient.

Definition 3.3. *Let (μ, f) be a measure-section pair over Ω with values in W. The* graph measure *$[\mu, f]$ is the Radon measure on W such that*

$$\int_W \psi \, d[\mu, f] = \int_\Omega \psi(f) \, d\mu$$

for every $\psi \in C_0^0(W)$.

Hutchinson pointed out that the definition of L^p-strong convergence can be rewritten in terms of graph measures: we have $(\mu_k, f_k) \to (\mu, f)$ in the L^p-strong sense if and only if $[\mu_k, f_k] \to [\mu, f]$ weakly* in $(C_0^0(W))^*$ and

$$\int_{\{y\in W\,:\,\delta(\varpi(y),x_0)\geq j \text{ or } |y|\geq j\}} |y|^p \, d[\mu_k, f_k](y) \to 0 \quad \text{as } j \to \infty$$

uniformly in k.

We need another result on the convergence of measure-section pairs. This is a further analogue of a well-known fact in the usual L^p-theory.

Proposition 3.1. *Let $p, q \in (1, \infty)$ and $r \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. For $k \in \mathbb{N}$, suppose that μ_k is a Radon measure on Ω and f_k, g_k are sections of W with $|f_k| \in L^p(\mu_k)$ and $|g_k| \in L^q(\mu_k)$. Furthermore, let μ be a Radon measure on Ω and f, g sections with $|f| \in L^p(\mu)$ and $|g| \in L^q(\mu)$. If (μ_k, f_k) converges L^p-strongly to (μ, f) and (μ_k, g_k) converges L^q-weakly to (μ, g), then $(\mu_k, \langle f_k, g_k\rangle)$ converges L^r-weakly to $(\mu, \langle f, g\rangle)$ in the vector bundle $\Omega \times \mathbb{R}$.*

Proof. It suffices to consider the case $r = 1$, as the uniform bound for the L^r-norms follows from Hölder's inequality. Furthermore, it suffices to prove the statement for a subsequence.

Let $\tilde{W} = \varpi^* W$ be the pull-back bundle over W. Consider the Radon measures $\tilde{\mu}_k = [\mu_k, f_k]$ on W and the sections $\tilde{g}_k = g_k \circ \varpi$ of $\tilde{W}$. We have

$$\int_W |\tilde{g}_k|^q \, d\tilde{\mu}_k = \int_\Omega |g_k|^q \, d\mu_k,$$

and the right hand side is uniformly bounded. By Theorem 3.1, we may assume (passing to a subsequence if necessary) that we have L^q-weak convergence of $(\tilde{\mu}_k, \tilde{g}_k)$. Let $(\tilde{\mu}, \tilde{g})$ be the L^q-weak limit. Then clearly $\tilde{\mu} = [\mu, f]$. We first want to show that $g(x) = \tilde{g}(f(x))$ for μ-almost every $x \in \Omega$.

Let ϕ be a continuous section of W with compact support. Then $\phi \circ \varpi$ is a continuous section of $\tilde{W}$. For every $j \in \mathbb{N}$, choose a cut-off function $\psi_j \in C_0^0(\mathbb{R})$ with $0 \leq \psi_j \leq 1$ and $\psi_j(s) = 1$ for $|s| \leq j$. Then we have

$$\begin{aligned} \int_\Omega \langle \phi, g\rangle \, d\mu &= \lim_{k\to\infty} \int_\Omega \langle \phi, g_k\rangle \, d\mu_k = \lim_{k\to\infty} \int_W \langle \phi \circ \varpi, \tilde{g}_k\rangle \, d\tilde{\mu}_k \\ &= \lim_{k\to\infty} \lim_{j\to\infty} \int_W \psi_j(|y|) \, \langle \phi(\varpi(y)), \tilde{g}_k(y)\rangle \, d\tilde{\mu}_k(y) \end{aligned} \tag{12}$$

by Lebesgue's convergence theorem. Moreover,

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_W \psi_j(|y|)\left\langle \phi(\varpi(y)), \tilde{g}_k(y)\right\rangle \, d\tilde{\mu}_k(y)$$
$$= \lim_{j\to\infty}\int_W \psi_j(|y|)\left\langle \phi(\varpi(y)), \tilde{g}(y)\right\rangle \, d\tilde{\mu}(y) = \int_\Omega \langle \phi, \tilde{g}\circ f\rangle \, d\mu.$$

Now we note that

$$\begin{aligned}\left|\int_W (\psi_j(|y|)-1)\left\langle \phi(\varpi(y)), \tilde{g}_k(y)\right\rangle \, d\tilde{\mu}_k(y)\right| &= \left|\int_\Omega (\psi_j\circ|f_k|-1)\langle \phi, g_k\rangle \, d\mu_k\right| \\ &\leq \int_{\{x\in\Omega\colon\, |f_k(x)|\geq j\}} |\langle \phi, g_k\rangle| \, d\mu_k \\ &\leq (\mu_k(\{x\in \operatorname{supp}\phi\colon\, |f_k(x)|\geq j\}))^{1/p}\|g_k\|_{L^q(\mu_k)}\sup_\Omega|\phi|.\end{aligned}$$

The right hand side converges to 0 uniformly in k as $j\to\infty$ by the strong convergence. Therefore the last step in (12) involves uniform convergence and we can interchange the limits. It follows that $g(x)=\tilde{g}(f(x))$ for μ-almost every $x\in\Omega$.

Similarly, for $\eta\in C_0^0(\Omega)$, we compute

$$\lim_{k\to\infty}\int_\Omega \eta\langle f_k, g_k\rangle \, d\mu_k = \lim_{k\to\infty}\lim_{j\to\infty}\int_W \psi_j(|y|)\eta(\varpi(y))\left\langle y, \tilde{g}_k(y)\right\rangle \, d\tilde{\mu}_k(y)$$

and

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_W \psi_j(|y|)\eta(\varpi(y))\left\langle y, \tilde{g}_k(y)\right\rangle \, d\tilde{\mu}_k(y)$$
$$= \int_\Omega \eta\langle f, \tilde{g}\circ f\rangle \, d\mu = \int_\Omega \eta\langle f, g\rangle \, d\mu.$$

Again we see that the limits can be interchanged, this time using the fact that

$$\lim_{j\to\infty}\int_{\{x\in\Omega\colon\, |f_k(x)|\geq j\}} |f_k|^p \, d\mu_k = 0$$

uniformly in k. Hence

$$\lim_{k\to\infty}\int_\Omega \eta\langle f_k, g_k\rangle \, d\mu_k = \int_\Omega \eta\langle f, g\rangle \, d\mu,$$

as required. □

Applying the results to functions in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and the weak second fundamental forms of their level sets, we obtain the following statement.

Corollary 3.1. *For $k\in\mathbb{N}$, let $u_k\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$, and let $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$. Suppose that $u_k\rightharpoonup u$ weakly in $L^1_{\mathrm{loc}}(\Omega)$ and $|Du_k|\overset{*}{\rightharpoonup}|Du|$ weakly* in $(C_0^0(\Omega))^*$. Furthermore, suppose that there exist weak second fundamental forms A_k of the level sets of u_k with*

$$\sup_{k\in\mathbb{N}}\int_\Omega |A_k|^2 \, d|Du_k| < \infty.$$

(i) Then there exists a weak second fundamental form A of the level sets of u.

(ii) Suppose that ν_k and ν are unit vector fields with

$$Du_k = |Du_k| \llcorner \nu_k \quad \textit{and} \quad Du = |Du| \llcorner \nu.$$

Moreover, let H be the (infinite-dimensional) vector bundle over Ω with fibre $C_0^0(T_x\Omega; \mathrm{End}(T_x\Omega))$ at $x \in \Omega$. Then there exists a subsequence $(k_\ell)_{\ell\in\mathbb{N}}$ such that for every continuous section ϕ of H with compact support,

$$\int_\Omega \langle A, \phi(\nu)\rangle \, d|Du| = \lim_{\ell\to\infty} \int_\Omega \langle A_{k_\ell}, \phi(\nu_{k_\ell})\rangle \, d|Du_{k_\ell}|. \tag{13}$$

Proof. Let $\psi \in C_0^0(\Omega)$. Consider first the measure-section pairs $(|Du_k|, \psi\nu_k)$ with values in $T\Omega$. Clearly we have L^2-weak convergence to $(|Du|, \psi\nu)$. Furthermore,

$$\int_\Omega \psi^2 |\nu|^2 \, d|Du| = \lim_{k\to\infty} \int_\Omega \psi^2 |\nu_k|^2 \, d|Du_k|.$$

Thus we obtain L^2-strong convergence by Theorem 3.1. Let ϕ be a continuous section of H with compact support. Define $f_k = \phi(\nu_k)$ and $f = \phi(\nu)$. Then we also have L^2-strong convergence of $(|Du_k|, f_k)$ to $(|Du|, f)$.

Now we consider the measure-section pairs $(|Du_k|, A_k)$. By Theorem 3.1, there exists a subsequence converging L^2-weakly to a limit $(|Du|, A)$, where A is a section of $\mathrm{End}(T\Omega)$. Using Proposition 3.1, we infer (13). Testing the equation with appropriate functions, we see that A is the weak second fundamental form of the level sets of u. □

4 Passing to the limit

We now use the results of the previous sections to prove Theorem 1.1. Consider again an open set $\Omega \subset \mathcal{N}$ such that $E = \mathcal{N}\backslash\Omega$ is non-empty and $\overline{E^\circ} = E$. For $p \in (1, 2]$, let $v_p \in \dot{W}^{1,p}(\mathcal{N})$ be a minimizer of the functional $F_p(v) = \|v\|_{\dot{W}^{1,p}(\mathcal{N})}$ among all $v \in \dot{W}_0^{1,p}(\mathcal{N})$ with $v \geq 1$ almost everywhere in E. Then we have

$$\operatorname{div}(|\nabla v_p|^{p-2}\nabla v_p) = 0 \quad \text{in } \Omega.$$

Furthermore, define $u_p = (1-p)\log v_p$, so that

$$\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = |\nabla u_p|^p \quad \text{in } \Omega.$$

In addition, we have

$$\operatorname{div}(|\nabla v_p|^{p-2}\nabla v_p) \leq 0 \quad \text{in } \mathcal{N}$$

and

$$\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) \geq |\nabla u_p|^p \quad \text{in } \mathcal{N}.$$

With the same estimates as in section 2, we find

$$\int_\Omega \eta^p |\nabla u_p|^p \, dV \leq p^p \int_\Omega |\nabla\eta|^p \, dV$$

for every $\eta \in C_0^\infty(\mathcal{N})$ with $\eta \geq 0$. We also know that $u_p = 0$ almost everywhere in E. As $\mathcal{N}$ is connected, we obtain a local uniform L^1-bound by the Poincaré inequality. Hence u_p is locally uniformly bounded in the BV-norm and there exist a sequence $p_k \searrow 1$ and a function $u \in \mathrm{BV}_{\mathrm{loc}}(\mathcal{N})$ such that $u_{p_k} \to u$ in $L^1_{\mathrm{loc}}(\mathcal{N})$. Clearly $u = 0$ in E and $u \geq 0$ in $\mathcal{N}$.

There is a Harnack inequality for p-harmonic functions. When we calculate the Harnack constant, we find that for every $x_0 \in \Omega$ there exist an $r > 0$ and a constant $c > 1$ such that

$$\sup_{x,y \in B_r(x_0)} \frac{v_p(x)}{v_p(y)} \leq c^{1/(p-1)}$$

for every p. These computations have been carried out for $\mathcal{N} = \mathbb{R}^n$ in another paper [9], and for other manifolds they are similar. Thus the oscillation of u_p is locally uniformly bounded. Hence in every connected component of Ω, either u_p is locally uniformly bounded, or $u_p \to \infty$ locally uniformly. The latter, however, is inconsistent with the local uniform L^1-bounds. Hence if we choose a bounded, open set $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega$, then we have

$$\limsup_{k \to \infty} \sup_{\Omega'} u_{p_k} < \infty.$$

Using Proposition 2.1 repeatedly, we see that the functions u_p are uniformly bounded in $W^{1,q}(\Omega')$ for every $q < \infty$. Hence $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $u_{p_k} \rightharpoonup u$ weakly in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$. By the Sobolev embedding theorem and the theorem of Arzelà-Ascoli, we also have local uniform convergence. With the same arguments as in a previous work [9, pp. 2249–2250] we conclude that u is a weak solution of the inverse mean curvature flow. These arguments also show that $|Du_{p_k}| \overset{*}{\rightharpoonup} |Du|$ weakly* in $(C_0^0(\Omega))^*$.

Next we examine the behaviour at the boundary. Suppose first that $x_0 \in \partial\Omega$ is a regular point. Fix a constant $K > 0$ such that $\mathrm{Ric} \geq -(n-1)K \langle \cdot, \cdot \rangle$ in $B_4(x_0)$. Fix $R_0 \in (0, 2]$ and $\epsilon \in (0, 1]$. Then there exist a point $y_0 \in E$ and a number $R \in (0, R_0/2]$ such that $B_R(y_0) \subset E$ and $\delta(x_0, y_0) \leq (1+\epsilon)R$. We now estimate u_p using a barrier function constructed by Kotschwar and Ni [7, Sect. 3].

To this end, define first

$$h(\rho) = \left(\rho e^{\rho\sqrt{K}}\right)^{n-1}$$

and

$$\phi_p(r) = \frac{\int_r^{R_0} (h(\rho))^{1/(1-p)}\, d\rho}{\int_R^{R_0} (h(\rho))^{1/(1-p)}\, d\rho}, \quad R \leq r \leq R_0.$$

Furthermore, let

$$w_p(x) = \phi_p(\delta(x, y_0)), \quad x \in B_{R_0}(y_0) \backslash B_R(y_0).$$

Kotschwar and Ni showed that w_p is p-subharmonic if R_0 is chosen sufficiently small. By construction, we have $w_p = 1$ on $\partial B_R(y_0)$ and $w_p = 0$ on $\partial B_{R_0}(y_0)$. Thus $v_p \geq w_p$ and $u_p \leq (1-p)\log w_p$ in $B_{R_0}(y_0) \backslash B_R(y_0)$. Now we estimate w_p in $B_{(1+2\epsilon)R}(y_0) \backslash B_R(y_0)$.

Consider $r \in (R, (1+2\epsilon)R)$. We have

$$\int_r^{R_0} (h(\rho))^{1/(1-p)}\, d\rho = \frac{R_0 - r}{R_0 - R} \int_R^{R_0} \left(h\left(\frac{R_0 - \rho}{R_0 - R} r + \frac{\rho - R}{R_0 - R} R_0 \right) \right)^{\frac{1}{1-p}} d\rho$$

by the substitution rule. Let $\rho \in [R, R_0]$ and set

$$\rho_* = \frac{R_0 - \rho}{R_0 - R} r + \frac{\rho - R}{R_0 - R} R_0.$$

Then we have $\rho_* \geq \rho$ and

$$\rho_* - \rho = \frac{r - R}{R_0 - R}(R_0 - \rho) \leq 4\epsilon R$$

and

$$\frac{\rho_*}{\rho} = \frac{R_0 - r}{R_0 - R} + \frac{(r-R)R_0}{(R_0 - R)\rho} \leq 1 + 2\epsilon.$$

Hence there exists a constant c such that

$$\frac{h(\rho_*)}{h(\rho)} \leq 1 + c\epsilon$$

for $R \leq r \leq (1+2\epsilon)R$ and $R \leq \rho \leq R_0$. Therefore,

$$\phi_p(r) \geq \frac{R_0 - r}{R_0 - R}(1 + c\epsilon)^{1/(1-p)}.$$

Since $B_{\epsilon R}(x_0) \subset B_{(1+2\epsilon)R}(y_0)$, it follows that $u \leq \log(1 + c\epsilon)$ in $B_{\epsilon R}(x_0)$. As ϵ was chosen arbitrarily, we have

$$\lim_{x \to x_0} u(x) = 0.$$

Now we claim that the set of all regular boundary points is dense in $\partial\Omega$. Let $x_0 \in \partial\Omega$ and $\rho > 0$. Because $\overline{E^\circ} = E$, there exists an interior point y_0 of E in $B_\rho(x_0)$. Choose $\theta > 0$ such that $B_\theta(y_0) \subset E \cap B_\rho(x_0)$. Let $\gamma : [0,1] \to \mathcal{N}$ be a parametrization of a shortest geodesic between x_0 and y_0, with $\gamma(0) = x_0$ and $\gamma(1) = y_0$. Define

$$s_0 = \inf \{ s \in (0,1] \colon B_\theta(\gamma(s)) \subset E \}.$$

Then the ball $B_\theta(\gamma(s_0))$ is contained in E but touches $\partial\Omega$ at some point, say $z_0 \in \partial\Omega$. Thus z_0 is a regular point, but by construction, we have $z_0 \in B_\rho(x_0)$. The claim follows. It also follows that

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = 0.$$

Finally we prove the required estimates for the second fundamental form. Let A_p be the section of $\mathrm{End}(T\mathcal{N})$ defined as in (10) for u_p. We have

$$\limsup_{p \searrow 1} \int_{\Omega'} |A_p|^2 |\nabla u_p| \, dV < \infty$$

for every precompact set $\Omega' \subset \Omega$ by Proposition 2.1. Hence we can apply Corollary 3.1. It follows that there exists a weak second fundamental form A of the level sets of u. Moreover, we may assume that (13) holds for the sequence u_{p_k}.

Now suppose that $u(x) \to \infty$ as $\text{dist}(x, E) \to \infty$. Consider the number T defined in part (vi) of the theorem. Let $t_0 > T$ and choose $t_1 \in (T, t_0)$. Then $u^{-1}([t_1, \infty)) \subset \Omega$. Choose $\psi \in C_0^\infty(t_1, \infty)$ with $\psi \geq 0$ and $\psi(t_0) = 1$ and with $0 \leq \psi' \leq 2/(t_0 - t_1)$ in $[t_1, t_0]$ and $\psi' \leq 0$ in $[t_0, \infty)$. Then we find

$$\int_\Omega \psi(u)e^{-u}|\nabla u||A|^2\,dV \leq \liminf_{k\to\infty} \int_\Omega \psi(u_{p_k})e^{-u_{p_k}}|\nabla u_{p_k}||A_{p_k}|^2\,dV.$$

Similarly,

$$\int_{u^{-1}([t_0,\infty))} \psi'(u)e^{-u}|\nabla u|^3\,dV \geq \limsup_{k\to\infty} \int_{u_{p_k}^{-1}([t_0,\infty))} \psi'(u_{p_k})e^{-u_{p_k}}|\nabla u_{p_k}|^3\,dV,$$

and

$$C := \limsup_{k\to\infty} \int_{u_{p_k}^{-1}((t_1,t_0])} \frac{2e^{-u_{p_k}}}{t_0 - t_1}|\nabla u_{p_k}|^3\,dV < \infty.$$

Thus inequality (11) implies

$$2\int_\Omega \psi(u)e^{-u}|\nabla u||A|^2\,dV - \int_{u^{-1}([t_0,\infty))} \psi'(u)e^{-u}|\nabla u|^3\,dV \leq C.$$

Now define

$$E_t = u^{-1}([0,t]).$$

Then for almost all $t \geq t_0$, the set E_t has finite perimeter and we can consider its reduced boundary $\partial^* E_t$. Moreover, by the coarea formula [1, Theorem 3.40], we can write the last inequality in the form

$$2\int_0^\infty \psi(t)e^{-t}\int_{\partial^* E_t} |A|^2\,d\sigma\,dt - \int_{t_0}^\infty \psi'(t)e^{-t}\int_{\partial^* E_t} |\nabla u|^2\,d\sigma\,dt \leq C$$

It follows that for almost all $\tau > t_0$,

$$e^{-\tau}\int_{\partial^* E_\tau} |\nabla u|^2\,d\sigma + 2\int_{t_0}^\tau e^{-t}\int_{\partial^* E_t} |A|^2\,d\sigma\,dt \leq C.$$

Hence u has all the properties stated in Theorem 1.1.

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